Euclidean division and commutativity in $(LO, +)$

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Outline

Goals:

i. Describe a connection between *additive commutativity* of linear orders and a generalized *Euclidean division algorithm*.
ii. Discuss how recognizing this connection helped us recently solve two problems posed by Tarski in 1956.

The results in this talk were obtained without the use of AI tools.

$+$ as concatenation of natural numbers

If we view each $n \in \mathbb{N}$ as an ordered set of n points,

$$\underbrace{(\bullet \quad \bullet \cdots \bullet)}_{n}$$

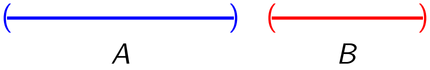
Then the sum $+$ on $\mathbb{N}$ can be viewed as a concatenation operation:

$$\underbrace{(\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet)}_{5} + \underbrace{(\bullet \quad \bullet)}_{2} = \underbrace{(\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \mid \bullet \quad \bullet)}_{7}$$

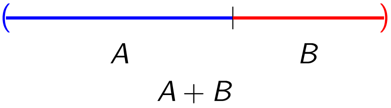
$+$ as concatenation of linear orders

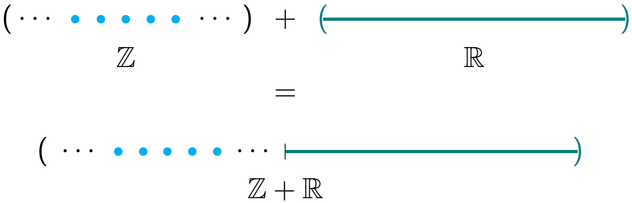
We can generalize the notion of sum-as-concatenation to the class of all linear orders.

Def: Given two linear orders A and B,



Their *sum* $A + B$ is the linear order obtained by placing a copy of B to the right of A.




$\mathbb{Z}$
$\mathbb{R}$
$=$
$\mathbb{Z} + \mathbb{R}$

Arithmetic in $(LO, +)$ vs. $(\mathbb{N}, +)$

Let LO denote the class of linear orders.

Big question: To what extent does the arithmetic of $(LO, +)$ resemble that of $(\mathbb{N}, +)$?

- Which arithmetic laws in $(\mathbb{N}, +)$ still hold in $(LO, +)$?

  e.g. associativity, commutativity, additive cancellation, archimedean comparability, Euclidean divisibility, etc.
- For those laws that fail, can we characterize when they hold?

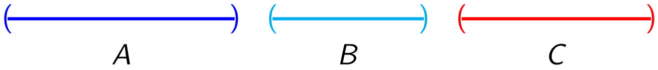
  focus in this talk: commutativity in $(LO, +)$ and its relation to Euclidean division.

Associativity in $(\mathbb{N}, +)$ and $(LO, +)$

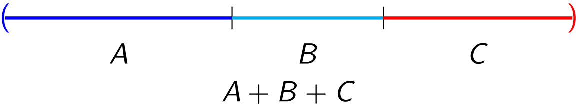
$(\mathbb{N}, +)$ satisfies the associativity law:

$$a + (b + c) = (a + b) + c.$$

We retain associativity in $(LO, +)$: for all $A, B, C \in LO$,



We have $(A + B) + C \cong A + (B + C)$.



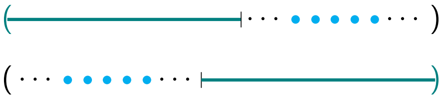
Commutativity in $(\mathbb{N}, +)$ and $(LO, +)$

$(\mathbb{N}, +)$ satisfies the commutativity law:

$$a + b = b + a.$$

Commutativity fails in $(LO, +)$ in general, e.g.

$$\mathbb{R} + \mathbb{Z} \ncong \mathbb{Z} + \mathbb{R}$$



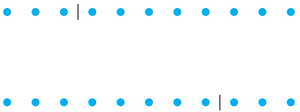
Commutativity in $(LO, +)$

Commutativity doesn't *always* fail in $(LO, +)$.

$A + B \cong B + A$ if ...

- $A = n$ and $B = m$ are natural numbers;
- More generally, there is a linear order C such that $A = nC$ and $B = mC$.

(Here, nC denotes the n-fold sum $C + C + \cdots + C$.)



A commutativity conjecture

Conjecture: $A + B \cong B + A$ if and only if $A \cong nC$ and $B \cong mC$ for some $C \in LO$ and $n, m \in \mathbb{N}$.

A counterexample: additive bi-absorption

$\mathbb{N}$ additively absorbs 1 on the left:

$$\underset{1}{\bullet} \; + \; \underset{\mathbb{N}}{\bullet\bullet\bullet\bullet\cdots} \;\cong\; \underset{\mathbb{N}}{\bullet|\bullet\bullet\bullet\bullet\cdots}$$

Symmetrically, $\mathbb{N}^*$ additively absorbs 1 on the right:

$$\underset{\mathbb{N}^*}{\cdots\bullet\bullet\bullet\bullet} \; + \; \underset{1}{\bullet} \;\cong\; \underset{\mathbb{N}^*}{\cdots\bullet\bullet\bullet\bullet|\bullet}$$

A counterexample: additive bi-absorption

Thus $\mathbb{N} + \mathbb{N}^*$ bi-absorbs 1:

$$\bullet | \bullet \;\bullet \;\bullet \;\cdots \qquad \cdots \;\bullet \;\bullet \;\bullet | \bullet$$

That is,

$$1 + (\mathbb{N} + \mathbb{N}^*) \cong (\mathbb{N} + \mathbb{N}^*) + 1 \cong \mathbb{N} + \mathbb{N}^*.$$

In particular,

$$1 + (\mathbb{N} + \mathbb{N}^*) \cong (\mathbb{N} + \mathbb{N}^*) + 1.$$

Tarski's commutativity conjecture

Conjecture: (Tarski; 1930s, unpublished) $A + B \cong B + A$ if and only if one of the following holds:

i. $A \cong nC$ and $B \cong mC$ for some $C \in LO$ and $n, m \in \mathbb{N}$,

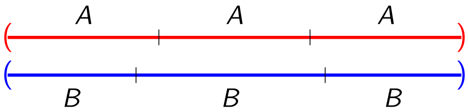
ii. $A + B \cong B + A \cong A$,

iii. $A + B \cong B + A \cong B$.

Euclidean division in $(LO, +)$

Finite division can be carried out in $(LO, +)$ in a strikingly general sense.

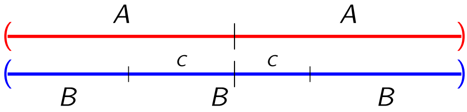
Cancellation theorem: (Lindenbaum) Suppose $nA \cong nB$ for some natural number $n \geq 1$. Then $A \cong B$.



Pf: Not immediate.

Euclidean division in $(LO, +)$

Division theorem: (Lindenbaum) Suppose $nA \cong mB$ for some natural numbers n and m with $\gcd(n, m) = 1$. Then there is a linear order C such that $A \cong mC$ and $B \cong nC$.



Pf: Surprisingly hard!

Some history

Many of the fundamental results about $(LO, +)$, including the cancellation and division theorems above, were proved by Lindenbaum.

A. Lindenbaum

Some history

Lindenbaum's results are stated *without proofs* in a book with Tarski (1926).

COMMUNICATION SUR LES

RECHERCHES DE LA THÉORIE

DES ENSEMBLES

Coauthored with Adolf Lindenbaum

13 (L). Si $\alpha . k = \beta . k$ et $k \neq 0$, alors $\alpha = \beta$.

14 (L). Si $\alpha . k = \beta . l$, où k et l sont deux nombres finis premiers entre eux, il existe un type ξ tel que $\alpha = \xi . l$ et $\beta = \xi . k$.

Some history

Proofs of Lindenbaum's results would not appear until 30 years later, after Lindenbaum's death, in Tarski's book *Ordinal Algebras* (1956).

ORDINAL ALGEBRAS

BY

ALFRED TARSKI

Professor of Mathematics,
University of California, Berkeley

WITH APPENDICES BY

CHEN CHUNG CHANG

Instructor in Mathematics, Cornell University

AND

BJARNI JÓNSSON

Assistant Professor of Mathematics, Brown University

N.H.P.C.

1956

Ordinal algebras

An *ordinal algebra*

$$(\mathfrak{A}, +, \Sigma, ^*, 0)$$

is a type of abstract structure generalizing $(LO, +)$ in which one can do "concatenation arithmetic."

It consists of

- a universe $\mathfrak{A}$ (whose elements $a \in \mathfrak{A}$ can be thought of as "segments"),
- a binary concatenation operation $+$,
- an $\mathbb{N}$-ary concatenation operation Σ,
- a unary reversal operation *,
- an identity element 0.

Ordinal algebras

The axioms for ordinal algebras were isolated by Tarski as the principles needed to prove Lindenbaum's results about $(LO, +)$.

(I) [POSTULATE OF ELEMENTARY SUMS]. $\sum_{\varkappa<0} a_\varkappa = 0$ *and* $\sum_{\varkappa<1} a_\varkappa = a_0$.

(II) [ASSOCIATIVE POSTULATE]. *If* $\mu < \omega$ *and* $\nu \leqslant \omega$, *then*

$$\sum_{\varkappa<\mu+\nu} a_\varkappa = \sum_{\varkappa<\mu} a_\varkappa + \sum_{\varkappa<\nu} a_{\mu+\varkappa}.$$

(III) [DIRECTED REFINEMENT POSTULATE]. *If* $\sum_{\varkappa<\omega} a_\varkappa = b + c$ *and* $c \neq 0$, *then there are two elements* d, e *and an ordinal* $\mu < \omega$ *such that* $\sum_{\varkappa<\mu} a_\varkappa + d = b$, $a_\mu = d + e$, *and* $e + \sum_{\varkappa<\omega} a_{\mu+1+\varkappa} = c$.

(IV) [REMAINDER POSTULATE]. *If* $a_\varkappa = b_\varkappa + a_{\varkappa+1}$ *for every* $\varkappa < \omega$, *then there is an element* c *such that* $a_\varkappa = \sum_{\lambda<\omega} b_{\varkappa+\lambda} + c$ *for every* $\varkappa < \omega$.

(V) [INVOLUTION POSTULATE]. $a^{**} = a$.

(VI) [DUAL ISOMORPHISM POSTULATE]. $(a+b)^* = b^* + a^*$. [3])

Tarski showed that Lindenbaum's results for $(LO, +)$, including the division theorem, hold in an arbitrary ordinal algebra $(\mathfrak{A}, +, \Sigma, ^*, 0)$.

Ordinal algebras

Results about $(LO, +)$ that follow from Tarski's ordinal algebra axioms can be thought of as "purely arithmetic."

That is, they can be proved only with reference to linear orders (the "segments" of LO) and the operations $+, \Sigma, ^*$.

Such proofs don't refer to underlying "points" in these segments.

Tarski's proof of the division theorem

Although Lindenbaum's division theorem is a direct generalization of Euclidean division in $(\mathbb{N}, +)$, Tarski's proof is involved, and not transparently related to the arithmetic of $(\mathbb{N}, +)$.

THEOREM 1.50 [EUCLID'S THEOREM]. *If $a \cdot \mu = b \cdot \nu$ where μ and ν are two relatively prime finite ordinals, then, for some c, $a = c \cdot \nu$ and $b = c \cdot \mu$.*

PROOF: I. We start with the special case $\mu = 2$, $\nu = 3$. Thus we have

(1) $$a \cdot 2 = b \cdot 3, \text{ i.e., } a + a = b + (b + b).$$

Hence, by 1.34, there is an element d such that either

(2) $$a + d = b \text{ and } a = d + b + b$$

or else

(3) $$a = b + d \text{ and } d + a = b + b.$$

In case (2) we have

$$a = (d + b) + a + d.$$

. . .

Our further argument in both cases (8) and (9) is entirely analogous. We restrict ourselves to the discussion of (8). By (6)–(8) we have

$$a = e + b = e + e + d = e + e + e + f = e \cdot 3 + f,$$

while (3), (7), and (8) give

$$a = b + d = e + d + d = e + e + f + e + f = e \cdot 3 + f \cdot 2.$$

Thus

$$a = e \cdot 3 + f = e \cdot 3 + f \cdot 2,$$

and hence, with the help of 1.13(ii),

$$a = a + f = e \cdot 3 + f \cdot 3 = (e + f) \cdot 3 = d \cdot 3. \tag{10}$$

...

And so on.

A new proof

Question: Is there a more transparent proof of Lindenbaum's division theorem?

Answer: (E. + Paul, 2026) Yes! In fact we found two.

- (High level) By adapting and extending an archimedean decomposition theory for groups G acting on linear orders X developed by McCleary (building on Holland, Hölder, Hahn, Conrad);
- (Direct) Via a generalized Euclidean algorithm for $(LO, +)$.

Characterizing commutativity in $(LO, +)$

In working on $(LO, +)$ and ordinal algebras, Tarski became very interested in characterizing the additively commuting pairs of linear orders. Recall his conjecture from above:

Tarski's Commutativity Conjecture: $A + B \cong B + A$ if and only if one of the following holds:

i. $A \cong nC$ and $B \cong mC$ for some $C \in LO$ and $n, m \in \mathbb{N}$,

ii. $A + B \cong B + A \cong A$,

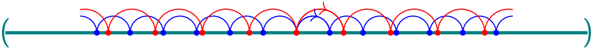
iii. $A + B \cong B + A \cong B$.

Tarski proved the conjecture holds over the class of scattered linear orders (and gave an ordinal algebra proof!).

Tarski's conjecture is false

Lindenbaum (1930s, unpublished) found a counterexample to Tarski's conjecture.

Later, Aronszajn (1950s) proved a structural characterization of the commuting pairs $A + B \cong B + A$. In so doing, he showed that Lindenbaum's counterexample is essentially the only possible one.



Roughly: Aronszajn showed such pairs A, B are obtained by replacing the points in a closed interval of $\mathbb{R}$ with linear orders in a way that respects a group of translations.

An arithmetic characterization of $A + B \cong B + A$?

Aronszajn's theorem is a *structural* characterization of the pairs $A + B \cong B + A$, not an *arithmetic* characterization.

There is no way to state Aronszajn's theorem in the language of ordinal algebras, much less ask whether it can be proved from the axioms for ordinal algebras.

This is in contrast to Tarski's commutativity conjecture, which can be stated in the language of ordinal algebras.

An arithmetic characterization of $A + B \cong B + A$?

Tarski noted this in his book!

obtained around 1930 but were not published; the counter-example of Lindenbaum was constructed in the same period. A general characterization of, and construction method for, arbitrary commutative couples of order types has recently been given by Aronszajn in [1]. From the main theorem in [1] the results concerning scattered and denumerable order types as well as Lindenbaum's counter-example can easily be derived. Aronszajn's results, however, cannot be formulated within the arithmetic of ordinal algebras.

An arithmetic characterization of $A + B \cong B + A$?

Jonssón, in a paper from the 80s, gives similar commentary:

There are still other properties that cannot be formulated in the language of ordinal algebras, e.g. properties that involve cardinalities of the structures, or involve addition with infinite index sets other than ω. A particularly interesting example of this arises in connection with the problem of characterizing those pairs of isomorphism types that commute with each other. The history of this problem is related in Tarski [1956], p. 80. Obviously $a \oplus b = b \oplus a$ holds if a and b are multiples of the same type,

$$a = \underline{m} \circ c \quad \text{and} \quad b = \underline{n} \circ c \quad ,$$

and also if one of them absorbs the other, both on the left and on the right, i.e., if either

$$a = (\omega \circ b) \oplus c \oplus (\omega \circ {*}b)$$

or

$$b = (\omega \circ a) \oplus c \oplus (\omega \circ {*}a) \quad .$$

Tarski proved that if either a or b is countable, and if $a \oplus b = b \oplus a$, then one of these three conditions must be satisfied. On the other hand, Lindenbaum constructed uncountable totally ordered sets for which this is not the case. Neither result has been published, but in Aronszajn [1952] the commuting pairs were completely characterized, and the earlier results derived as corollaries. This characterization is rather involved, and will not be described here. It does involve summations over non-denumerable sets of real numbers.

An arithmetic characterization of $A + B \cong B + A$?

Questions:

1. Is there an arithmetic characterization of $A + B \cong B + A$ (i.e., one that can be stated in the language of ordinal algebras)?
2. If so, is there an arithmetic proof (i.e., can it be proved from the axioms of ordinal algebras)?

Answers: Yes and yes!

An arithmetic characterization of $A + B \cong B + A$

Theorem (E. + Paul, 2026): $A + B \cong B + A$ if and only if one of the following conditions holds.

i. $\mathbb{N}A \cong \mathbb{N}B$ and $\mathbb{N}^*A \cong \mathbb{N}^*B$.
ii. $A + B \cong B + A \cong A$,
iii. $A + B \cong B + A \cong B$.

Theorem can be viewed as a "corrected" version of Tarski's original conjecture: Tarski's condition $A \cong nC$ and $B \cong mC$ is a strict strengthening of (i.).

Our '26 paper proves the theorem over LO. In forthcoming paper we show it holds in an arbitrary ordinal algebra.

Tarski's problem

There is exactly one explicitly stated open problem in all of *Ordinal Algebras*.

Tarski's Commuting Sum Problem: Suppose $A, B \in LO$ and there are natural numbers $n, m, k, l \geq 1$ such that

$$nA + mB \cong kB + lA.$$

Does it follow that $A + B \cong B + A$?

Tarski does not explain his interest in the problem, but in hindsight it is natural and closely connected to Lindenbaum's division results.

Tarski's and Chang's partial results

$$nA + mB \cong kB + lA \Rightarrow A + B \cong B + A?$$

Tarski showed the answer is yes if either $n = l$ or $m = k$.

- e.g. $A + 2B \cong 2B + A$ implies $A + B \cong B + A$.

C. C. Chang, with effort, extended Tarski's result to show answer is yes if both $n \leq l$ and $m \leq k$.

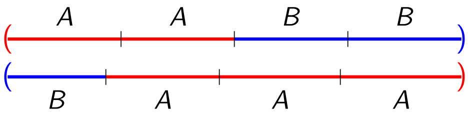
- e.g. $2A + 2B \cong 3B + 3A$ implies $A + B \cong B + A$.

Remaining case

$$nA + mB \cong kB + lA \Rightarrow A + B \cong B + A?$$

Tarski and Chang's results leave open when $n < l$ and $m > k$.

- e.g. does $2A + 2B \cong B + 3A$ imply $A + B \cong B + A$?



Remaining case

Concerning this remaining case, Chang wrote:

$\pi < \varrho$. The discussion of this remaining problem seems to present considerable difficulties. A.3 contains the affirmative solution of

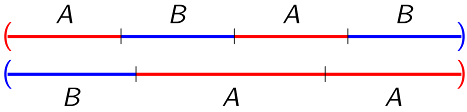
A general solution

The answer to Tarski's question is yes in every case. In fact, something more general is true.

Theorem (E. + Paul, 2026) Suppose A and B are linear orders and there is an isomorphism

$$C_0 + C_1 + \cdots + C_{n-1} \cong D_0 + D_1 + \cdots + D_{m-1}$$

such that $C_i, D_j \in \{A, B\}$, $C_0 = D_{m-1} = A$, and $D_0 = C_{n-1} = B$. Then $A + B \cong B + A$.

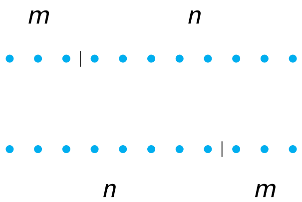


Euclidean division as a consequence of commutativity

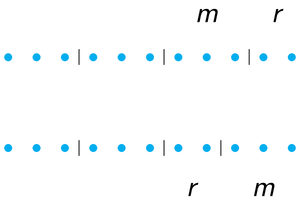
Key observation:

- The isomorphism $A + B \cong B + A$ is a setup for a 1D dynamical system (generalized interval exchange) that executes a Euclidean algorithm.
- A run of algorithm yields a shared decomposition of $A + B$ and $B + A$ as the same ordered sum of remainders.

$m + n = n + m$

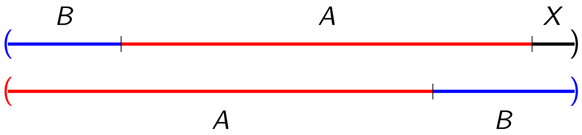


$m + n = n + m$

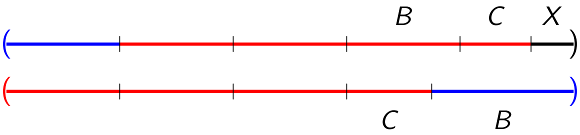


$$
\begin{aligned}
m + n &= km + (m + r) \\
n + m &= km + (r + m) \\
\\
m + r &= r + m
\end{aligned}
$$

$B + A \leqslant_{initial} A + B$



$B + A \leqslant_{initial} A + B$



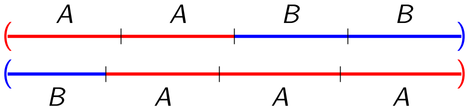
$$\begin{aligned} B + A &\cong kB + (B + C + X) \\ A + B &\cong kB + (C + B) \end{aligned}$$

$$B + C + X \cong C + B$$

Solving Tarski

How does this generalized Euclidean algorithm help solve Tarski's problem?

Suppose



Then $A + B$ and $B + A$ are aligned on the left, hence one is initial in the other.

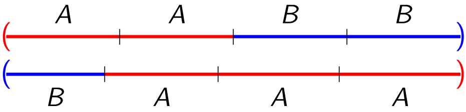
Divide from the left

- Suppose $B + A \leqslant_{init} A + B$, i.e. $A + B \cong B + A + X$.
- Run the algorithm to get decompositions

$$\begin{aligned} A + B &\cong k_1 A_1 + k_2 A_2 + \cdots + R \\ B + A &\cong k_1 A_1 + k_2 A_2 + \cdots + R'. \end{aligned}$$

Align on the right

- Notice that



  also gives that $A + B$ and $B + A$ are aligned *on the right*.
- Using this (and some structural jazz), we can show $R \cong R'$, i.e. $A + B \cong B + A$.

Thank you!